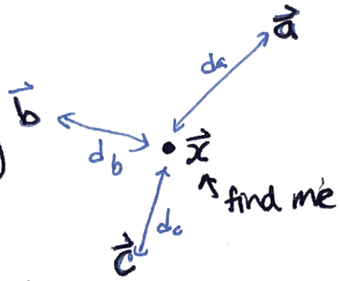
M3- Lecture 3



Summary: Last Lecture

Multiple Beacons A, B, C sing different (orthogonal) songs at once. Their positions known.

Each has a shift of $\{N_a, N_b, N_c\}$ in # samples

↳ convert to distance by $d_a = N_a \cdot T_s \cdot V$ → distance!

(# samples, Time/Sample, speed (m/s))

Received signal at $\vec{x}$ is $\vec{r} = \vec{S}_a^{(N_a)} + \vec{S}_b^{(N_b)} + \vec{S}_c^{(N_c)}$

(shift; song of beacon a; beacon index)

How to find shifts from recieved signal? Cross-correlate $\vec{r}$ with $\vec{S}_a, \vec{S}_b, \vec{S}_c$

$\vec{\rho}_{\vec{r},\vec{s}_a} = C_{\vec{s}_a} \cdot \vec{r}$

(cross-correlation of $\vec{r}$ with $\vec{s}_a$; circulant matrix of $\vec{s}_a$)

Find peak/max. value of $\vec{\rho}_{\vec{r},\vec{s}_a}$

↳ index of max. value is shift ~~a~~ N_a

* Then, repeat for $\vec{S}_b, \vec{S}_c$

$= C_{\vec{s}_a}\left[\vec{S}_a^{(N_a)} + \vec{S}_b^{(N_b)} + \vec{S}_c^{(N_c)}\right]$

$= \underbrace{C_{\vec{s}_a}\vec{S}_a^{(N_a)}}_{\text{want large @ } N_a} + \underbrace{C_{\vec{s}_a}\vec{S}_b^{(N_b)} + C_{\vec{s}_a}\vec{S}_c^{(N_c)}}_{\text{Want small}}$

cross-corr. $\vec{s}_a$ with $\vec{s}_b$ shifted → should not be corr.

<u>Want</u>: $\vec{\rho}_{\vec{r}\vec{s}_a}[m]$ to be max. @ $m = N_a$

mth element of corr. vector is just an inner product.

$\vec{\rho}_{\vec{r}\vec{s}_b}[j]$ to be max. @ $j = N_b$

$\vec{\rho}_{\vec{r}\vec{s}_c}[i]$ to be max. @ $i = N_c$

~~Received signals:~~

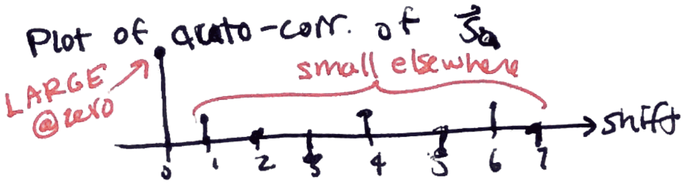
<u>Conditions</u>

1) $\vec{S}_a$ auto-corr @ non-zero shifts ≪ than auto-corr @ zero

2) $\vec{\rho}_{\vec{s}_a\vec{s}_b}$ very small (almost orthogonal)

Plots of cross-corr. $\vec{s}_a$ and $\vec{s}_b$:

Plot of auto-corr. of $\vec{s}_a$



$\boxed{\text{How should i generate songs?}}$

e.g. by fair coin flips (CS 70)

$$\|\vec{x}-\vec{a}\|^2 = d_a^2 \rightarrow \|\vec{x}\|^2 - 2\vec{a}^T\vec{x} + \|\vec{a}\|^2 = d_a^2 \quad (1)$$

← solve? will give circle, nonlinear! Need more info...

$$\|\vec{x}-\vec{b}\|^2 = d_b^2 \rightarrow \|\vec{x}\|^2 - 2\vec{b}^T\vec{x} + \|\vec{b}\|^2 = d_b^2 \quad (2)$$

$$\|\vec{x}-\vec{c}\|^2 = d_c^2 \rightarrow \|\vec{x}\|^2 - 2\vec{c}^T\vec{x} + \|\vec{c}\|^2 = d_c^2 \quad (3)$$

To get rid of non-linear $\|\vec{x}\|^2$ term, try:

(2) − (1) $\quad -2\vec{b}^T\vec{x} + 2\vec{a}^T\vec{x} + \|\vec{b}\|^2 - \|\vec{a}\|^2 = d_b^2 - d_a^2$

(3) − (1) $\quad -2\vec{c}^T\vec{x} + 2\vec{a}^T\vec{x} + \|\vec{c}\|^2 - \|\vec{a}\|^2 = d_c^2 - d_a^2$

Write as matrix-vector multiply (LINEAR!)

$$\begin{bmatrix} (\vec{a}-\vec{b})^T \\ (\vec{a}-\vec{c})^T \end{bmatrix} \begin{bmatrix} \vec{x} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(d_b^2 - d_a^2) - \frac{1}{2}(\|\vec{b}\|^2 - \|\vec{a}\|^2) \\ \frac{1}{2}(d_c^2 - d_a^2) - \frac{1}{2}(\|\vec{c}\|^2 - \|\vec{a}\|^2) \end{bmatrix}$$

$$A \quad \vec{x} = \vec{b}$$

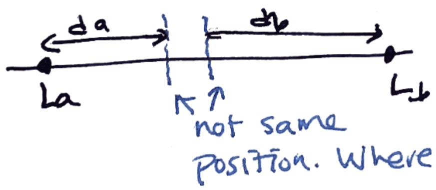
so $\vec{x} = A^{-1}\vec{b}$

* solves when $\vec{a}-\vec{b}$ and $\vec{a}-\vec{c}$ are linearly independent! (i.e. not collinear)
→ if dependent, only 1 eqn (of a line) ⇒ ∞ sol!

$\boxed{\text{Imperfect Measurements:}}$ what if 3 circles don't have 1 perfect intersection?

* "inconsistent meas." case from M1

e.g. 1D case



solutions:

- take more measurements of same kind. + average?
- take more 'diverse' measurements. ← How to average?
  different (e.g. more beacons)
- weight meas. by how accurate you think they are

Example: Estimate $\hat{x}$

$e_1(\hat{x}) = |a_1\hat{x} - b_1|$ error for meas. 1

$e_2(\hat{x}) = |a_2\hat{x} - b_2|$

maybe minimize $e_{abs}(\hat{x}) = e_1(\hat{x}) + e_2(\hat{x})$
$= |a_1\hat{x} - b_1| + |a_2\hat{x} - b_2|$

or minimize square: $e^2(\hat{x}) = e_1^2(\hat{x}) + e_2^2(\hat{x})$
$= |a_1\hat{x} - b_1|^2 + |a_2\hat{x} - b_2|^2$

use this! → $= (a_1\hat{x} - b_1)^2 + (a_2\hat{x} - b_2)^2$

How to minimize? Take derivative!

$$\frac{\partial}{\partial \hat{x}}\left(e^2(\hat{x})\right) = \frac{d}{d\hat{x}}\left[(a_1\hat{x}-b_1)^2\right] + \frac{d}{d\hat{x}}(a_2\hat{x}-b_2)^2 = 0$$

$$2a_1(a_1\hat{x}-b_1) + 2a_2(a_2\hat{x}-b_2) = 0$$

$$a_1^2\hat{x} + a_2^2\hat{x} - a_1b_1 - a_2b_2 = 0$$

$$\boxed{\hat{x} = \frac{a_1b_1 + a_2b_2}{a_1^2 + a_2^2}}$$

Does it make sense?

looks lik inner prod!

looks like norm²!

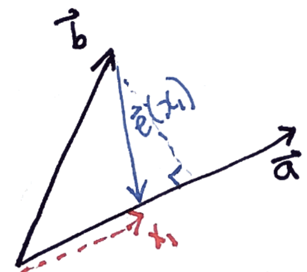
Together, looks like projection!

$$\hat{x} = \frac{\langle \vec{a}, \vec{b}\rangle}{\|\vec{a}\|^2}$$

How is min. error related to projection?

Error should be orthogonal to vector space!

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$



$\|x_1\| = \|\vec{a}x_1 - \vec{b}\|$

soln is restricted to be along $\vec{a}$ dir.

$\hat{\vec{b}} = \perp$ proj of $\vec{b}$ onto $\vec{a}$

$$= \frac{\langle \vec{a}, \vec{b}\rangle}{\|\vec{a}\|^2}$$

orthogonal projection onto allowed space will ALWAYS be one with smallest error!

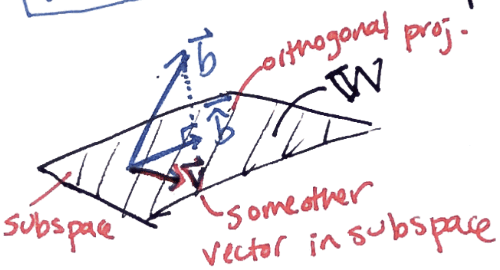
PROOF!

Assume I have some subspace $\mathbb{W} \in \mathbb{R}^N$, and some vector $\vec{b} \notin \mathbb{W}$ but $\vec{b} \in \mathbb{R}^N$

subspace   vector   not in subspace $\mathbb{W}$

$\hat{\vec{b}}$ = orthogonal proj. of $\vec{b}$ onto $\mathbb{W}$ ← Prove it's the proj. w smallest error.

How? Let's draw a picture.



Want to minimize $\|\vec{v} - \vec{b}\|$

or, $\hat{\vec{b}} = \underset{\vec{v}}{\arg\min} \|\vec{v} - \vec{b}\| \xrightarrow[\text{equiv.}]{} \|\vec{b} - \vec{v}\| > \|\vec{b} - \hat{\vec{b}}\|$

means the value of $\vec{v}$ at wich we have min. $\|-\|$

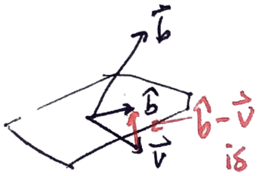
How to prove this?

start w/ error vector: $\vec{b}-\vec{v} = \vec{b}-\vec{v}+\hat{\vec{b}}-\hat{\vec{b}}$

want $\hat{b}$ somewhere... $= (\vec{b}-\hat{\vec{b}}) + (\hat{\vec{b}}-\vec{v})$

$$\|\vec{b}-\vec{v}\|^2 = (\vec{b}-\vec{v})^T(\vec{b}-\vec{v})$$

$$= \left[(\vec{b}-\hat{\vec{b}}) + (\hat{\vec{b}}-\vec{v})\right]^T \left[(\vec{b}-\hat{\vec{b}}) + (\hat{\vec{b}}-\vec{v})\right]$$



$$= \|\vec{b}-\hat{\vec{b}}\|^2 + \|\hat{\vec{b}}-\vec{v}\|^2 + 2(\vec{b}-\hat{\vec{b}})^T(\hat{\vec{b}}-\vec{v})$$

$\hat{b}-\vec{v}$ is in subspace

so $\|\vec{b}-\vec{v}\|^2 = \|\vec{b}-\hat{\vec{b}}\|^2 + \|\hat{\vec{b}}-\vec{v}\|^2$

>0

$\|\vec{b}-\vec{v}\|^2 > \|\vec{b}-\hat{\vec{b}}\|^2$ so $\hat{b}$ perpendicular is min. error sol'n!

$\vec{v} \in W$, $\vec{v} \neq \hat{\vec{b}}$ error ⊥ subspace means EVERY vector

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I have $\underset{m\times n}{A}\,\underset{n\times 1}{\vec{x}} = \underset{m\times 1}{\vec{b}}$ (A no longer square!)

measurements $= m$
unknowns $= n$
OVERDETERMINED
$m > n$
"extra measurements"

Extra measurements may not be redundant if they contain noise! (or error)

Two cases: 1) $\vec{b} \in \text{span}(A)$
2) $\vec{b} \notin \text{span}(A)$ ← what do I do?

Want $\vec{x}$ to minimize $\|A\vec{x}-\vec{b}\|$

what is subspace? $\text{span}(A) = \text{span}\{\vec{a}_1, \vec{a}_2, \dots \vec{a}_n\}$
what is measurement? $\vec{b}$

So $\hat{\vec{b}} = A\vec{x}$ orthogonal proj. of $\vec{b}$ onto W

$\vec{b}-\hat{\vec{b}} \perp W$ recall: all cols of A are in subspace, so:

$$\left.\begin{array}{l} \vec{b}-\hat{\vec{b}} \perp \vec{a}_1 \to \vec{a}_1^T(\vec{b}-\hat{\vec{b}}) = 0 \\ \vec{b}-\hat{\vec{b}} \perp \vec{a}_2 \to \vec{a}_2^T(\vec{b}-\hat{\vec{b}}) = 0 \\ \quad\vdots \qquad\qquad\qquad \vdots \\ b-\hat{\vec{b}} \perp \vec{a}_n \to \vec{a}_n^T(\vec{b}-\hat{\vec{b}}) = 0 \end{array}\right\} \begin{bmatrix} \vec{a}_1^T \\ \vec{a}_2^T \\ \vdots \\ \vec{a}_n^T \end{bmatrix} (\vec{b}-\hat{\vec{b}}) = 0$$

$$A^T(\vec{b}-\hat{\vec{b}}) = 0$$

$$A^T\hat{\vec{b}} = A^T\vec{b}$$

$$(A^TA)\vec{x} = A^T\vec{b}$$

Least-squares solution to $Ax=b$ $\boxed{\vec{x} = (A^TA)^{-1}A^T\vec{b}}$ ←

iff A^TA invertible